Compound Determinants.

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If a determinant of the order h, say A_h, be bordered by p rows and columns, thus making a determinant of the order $h+p$, then the rows and columns forming the border we will call a *gnomon of the order* p, and the determinant of the order $h+p$, obtained by putting on A_h a gnomon of the order p, we will call a p^{th} *gnomonic of* A_h or a *gnomonic of* A_h of the order $h+p$, and the principal diagonal of the square array at the intersection of the rows and columns forming the gnomon we will call the *principal diagonal of the gnomon.*

If from a determinant A_h of the order h we strike out p rows and columns leaving a minor of the order $h-p$, the rows and columns struck out will be called an *aphaereton* (the part which is taken away) *of the order* p, and the principal diagonal of the square array at the intersection of the rows and columns forming the aphaereton will be called the *principal diagonal of the aphaereton.*

It may happen that the very same thing will be designated by the name gnomon at one time and aphaereton at another time, but the point of view is different in the two cases; thus in Fig. 1, *dcgfbe* is a gnomon considered in reference to *ab* as a given determinant, but an aphaereton considered in reference to *ac* as a given determinant.

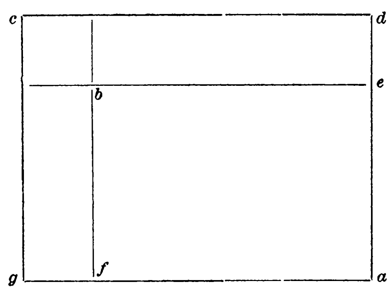

Fig. 1.

Now in Fig. 1, suppose *ab* or A_h to contain h, and *bc* to contain s rows and columns, so that *ac* contains $h+s$ $(=n)$ rows and columns. If we form a gnomon of A_h of the order p from p of the rows and columns intersecting in *bc*, and another gnomon of the order $q(=s-p)$ from the q remaining rows

and columns intersecting in bc, these two gnomons will be called *complementary* gnomons.

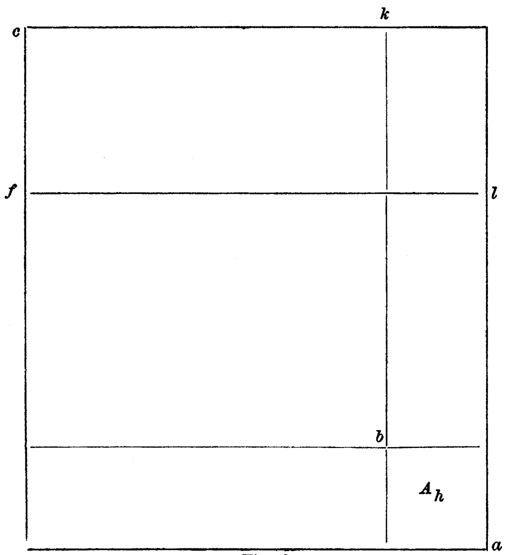

Fig. 2.

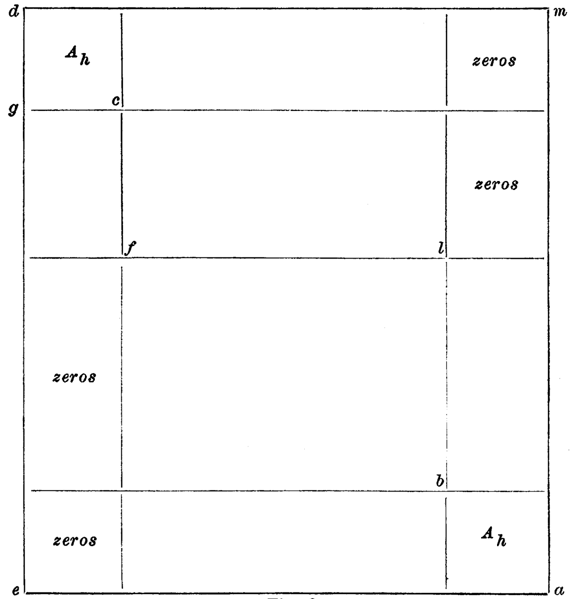

Fig. 3.

In Fig. 2, let ac be a determinant of the order n, say A_n, and select a minor ab or A_h of the order h. From this determinant ac we can form another as follows. Rewrite the last h rows of ac above the first row and the last h columns before the first column, then fill out the vacant upper left-hand corner with the determinant A_h, and finally change into zeros all the elements found at the same time in any of the first $h+p$ rows and the last h columns (*i. e.* in the rectangle lm in Fig. 3), and also all those elements found at the same time in any of the lower $n-p$ rows and the first h columns (*i. e.* in the rectangle ef in Fig. 3).

We have now the determinant ad, Fig. 3, of the order $n+h$, and if we add the first column to the $(n+1)^{\text{th}}$, the second to the $(n+2)^{\text{th}}$, etc., and then subtract the $(n+1)^{\text{th}}$ row from the first row, the $(n+2)^{\text{th}}$ from the second, etc., it is evident that the determinant $ad = A_h . A_n$, for it will then be the determinant A_n bordered by h rows and columns of which the square cd is the determinant, A_h and all the elements in the rectangle cm are zeros.

But the determinant ad, before any combination of rows and columns, is equal to the sum of all minors of the order $h+p$ formed from a given selection of $h+p$ rows each into its complementary minor. As a fixed selection of rows take the first $h+p$ rows, *i. e.* all those above the line fl, Fig. 3. Now any one

of these minors of the order $h+p$ which does not contain all the first h columns will have for its complementary minor a determinant containing a column of zeros, and therefore the product of the two minors must vanish. Also any one of these minors of the order $h+p$ which contains either of the last h columns will vanish from containing a column of zeros. Therefore, in applying Laplace's theorem to this case, the only significant terms are those wherein minors formed from the first $h+p$ rows *necessarily* contain all the first h columns and the complementary minors *necessarily* contain all the last h columns. The sign being properly determined,* all these latter described minors of the order $n-p$ are gnomonics of A_h of the order $n-p$, wherein each gnomon contains all the rows of the rectangle bf whether we consider Fig. 2 or Fig. 3.

Now consider any one of the non-vanishing minors of the order $h+p$ formed above the line fl, Fig. 3, and in that minor pass the first h columns bodily over the remaining p columns and then the first h rows bodily over the remaining p rows, and we have a determinant obtainable from Fig. 2 by putting upon A_h a gnomon of the p^{th} order whose rows and columns intersect in fk.†

From this we draw the important conclusion that if we form all possible gnomonics of A_h of the order $h+p$, wherein the gnomons used are formed from the rows and columns intersecting in fk, and each of these gnomonics be multiplied by the gnomonic of the order $n-p$ formed by putting upon A_h the complementary gnomon, the sum of all these products $= A_h . A_n$.

If one or more of the rows below the line fl should be identical with one or more of the rows above the line fl, then the sum of the products of determinants formed as above will evidently vanish; but we can arrive at this same set of determinants without supposing any two of the rows of A_n identical as follows. Border A_h in every possible way with gnomons of the order p from a fixed selection of p rows, and also in every possible way from a fixed selection of $(n-h-p)$ rows wherein the second fixed selection contains one or more rows of the first selection.

Therefore if we form all possible gnomonics of A_h of the order $h+p$ and also all possible gnomonics of A_h of the order $n-p$ wherein the gnomons used

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* Consider p^{th} and $(n-h-p)^{\text{th}}$ gnomonics of A_h formed by putting upon A_h complementary gnomons, then to all p^{th} gnomonics or gnomonics of the order $h+p$ give the sign $+$ and to all $(n-h-p)^{\text{th}}$ gnomonics give the sign $+$ or $-$ according as the principal diagonal of the gnomon used in forming the gnomonic into the principal diagonal of its complementary gnomon gives a positive or negative term of bc.

† This would not be the case if it were not for the fact that the elements in fg, Fig. 3, are the same as those in lk, in Fig. 2.

in forming the first set of gnomonics are taken from a fixed selection of p rows and those used in forming the second set of gnomonics are taken from a fixed selection of $n-h-p$ rows, which selection contains one or more rows of the first selection, and if each gnomonic of the first set is multiplied by that one of the second set in which the *gnomons* used in forming the two gnomonics have no two *columns* the same; then the sum of all these products must be zero.

Let us now form a compound determinant whose elements are p^{th} gnomonics of A_h, the various gnomons being taken from the rows and columns intersecting in bc. Suppose the elements of this determinant are denoted by $p_{r,s}$ and the whole determinant by P.

Also form another compound determinant whose elements are $(n-h-p)^{\text{th}}$ gnomonics of A_h, the gnomons being the complements of those used in forming the *corresponding* elements of P. Denote the elements of this determinant by $q_{r,s}$ and the whole determinant by Q.

From what has been shown above, it follows that P and Q are so related that when the elements of a row of P are multiplied by the corresponding elements of the *same* row of Q, the sum of the products $=A_h.A_n$ or $\sum^s p_{r,s}.q_{r,s}=A_h.A_n$, and if the elements of any row of P be multiplied by the corresponding elements of a *different* row of Q, the sum of the products $=0$ or $\sum^s p_{r,s}.q_{r',s}=0$.

Determinants possessing these two properties I shall call reciprocal. Evidently this definition includes the ordinary reciprocal determinants formed from complementary minors of a given determinant.

P and Q are therefore reciprocal determinants, and when multiplied together by the ordinary rule it follows at once that

$$P.Q=A_h^{(n-h)_p}.A_n^{(n-h)_p},$$

where $(n-h)_p$ stands for $\dfrac{(n-h)!}{p!\,(n-h-p)!}$ or the number of combinations of $n-h$ things taken p at a time.

From this value for the product $P.Q$ it follows that

$$P=x.A_h^s.A_n^{s'} \text{ and}$$

$$Q=\frac{1}{x}.A_h^{(n-h)_p-s}.A_n^{(n-h)_p-s'}.$$

Now when p changes into $n-h-p$, P becomes Q.* Therefore, whatever x is

* Q has the same value as if all its signs were $+$, for in every row the signs are the same as in the first row or every sign is reversed, and if we change the signs of the elements in every row which begins with a $-$ sign, then the signs in every row are the same as in the first row, and then changing the signs of all the columns which have $-$ signs, we make the same number of changes of sign of columns as we before made of rows, since originally the number of $-$ signs in the first row were the same as in the first column. The determinant has thus been multiplied by -1 an even number of times and the final determinant has all its signs $+$.

it must change into its reciprocal when $n-h-p$ is written for p. Also s and s' change into $(n-h)_p - s$ and $(n-h)_p - s'$ respectively. Moreover, when $h=0$, P becomes the determinant of minors of the p^{th} order formed from A_n and we know that then $P = A_n^{(n-1)_{p-1}}$, and when $p=1$ it can easily be proved* that then $P = A_h^{n-h-1}.A_n$, and when $h=n-1$, $P=A_n$.

From all these results we are led to write $x=1$, $s=(n-h-1)_p$, and $s'=(n-h-1)_{p-1}$.†

Hence $$P = A_h^{(n-h-1)_p}.A_n^{(n-h-1)_{p-1}}$$
and $$Q = A_h^{(n-h-1)_{p-1}}.A_n^{(n-h-1)_p}$$
since $$(n-h)_p - (n-h-1)_p = (n-h-1)_{p-1}.$$

In exactly the same manner as in ordinary reciprocal determinants it may be shown that if a minor of P of the order m be represented by P_m, we have

$$P_m = A_h^{m-(n-h-1)_{p-1}}.A_n^{m-(n-h-1)_p}.Q_{m'}$$

where $Q_{m'}$ is the complementary minor of Q; and if a minor of Q of the order m be represented by Q_m, we have

$$Q_m = A_h^{m-(n-h-1)_p}.A_n^{m-(n-h-1)_{p-1}}.P_{m'}$$

where $P_{m'}$ is the complementary minor of P. Gnomonics of A_h are minors of A_n, and the determinant above represented by P may be considered as the determinant of minors of A_n of the order $h+p$, which result from striking out, in every possible way, $n-h-p$ of the rows and columns intersecting in bc.

The determinant Q may also be considered as the determinant of minors of A_n, formed by striking out in every possible way p of the rows and columns intersecting in bc.

Now, since P and Q are reciprocal determinants, we have here an extension of the notion of reciprocal determinants, formed from *minors* of a given determinant, and these reduce to the ordinary reciprocal determinants when the striking out of a certain number of rows and columns is done in every possible way *throughout the whole determinant* A_n.

By making $h=0$ in the above expressions for P, Q, P_m, Q_m, and remembering that a determinant of the zero order $=1$, and also that $0!=1$, and

* See page 169.

† This inferring the form of a function from several particular values may be open to some objection as a proof, but it seems to me to be as free from objection as Picquet's method as given in Scott's Determinants, page 65, wherein an equation is assumed true which we know to be true for a *single* value of one of the letters entering the equation. The theorem was discovered by Professor Sylvester, but I have never seen his proof.

therefore $r_0 = 1$, we have the well-known values of these expressions given in most books on the subject.

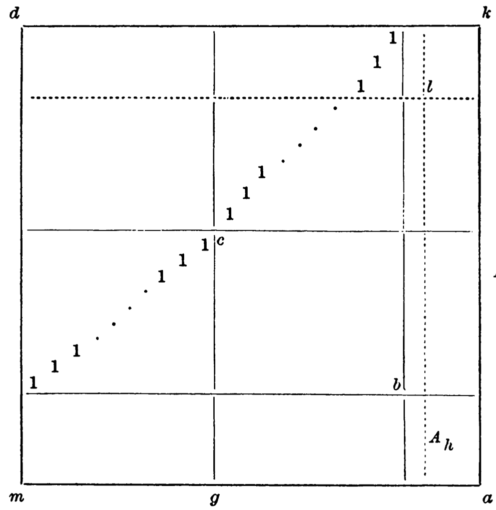

Fig. 4.

From the determinant ac, Fig. 4, of the order n, say A_n, let us form the determinant ad of the order $2n - h$ by bordering ac with $n - h$ rows and columns, as in Fig. 4, wherein all the elements of the gnomon are zeros except those marked 1 in the figure. It is evident that this determinant $ad = \pm$ the determinant ab which is of the order h and will be designated by A_h; the upper or lower sign being used according as $n - h$ is even or odd.

Let us now form the determinant of first minors of ad, and of the resulting determinant take that minor of the order $n - h$ which contains all the elements obtained by striking out in every possible way one of the rows and columns intersecting in cd.

If we call this minor D_{n-h} we have by a well-known theorem (Salmon's Higher Algebra, page 29, Art. 33)

$$D_{n-h} = A_h^{n-h-1} \cdot A_n .$$

But the elements of D_{n-h} are all possible first gnomonics of A_h, where the gnomons used are formed from the rows and columns intersecting in bc, and is the very determinant that we have before called P_m, when $p = 1$; and here we have the easy proof spoken of on page 168.

Now we know that if $ad = 0$ the first minors of any row are proportional to those of any other row (Salmon's Higher Algebra, page 29, ex. 1). Therefore if $A_h = 0$ the first gnomonics in any row are proportional to those in any

other row of the determinant of all first gnomonics, or the determinant of all first gnomonics, and also any minor above the first order vanishes.

Instead of first we may form p^{th} minors of ad by striking out p rows and columns, and it is evident that every p^{th} minor formed by removing an aphaereton of the p^{th} order whose rows and columns intersect in cd reduces to a p^{th} gnomonic of A_h, formed by bordering A_h with a gnomon of the p^{th} order whose rows and columns intersect in bc.

Thus we have a means of passing from theorems concerning minors to those concerning gnomonics, but to do so we should always look upon a minor as formed by *taking away an aphaereton* from a given determinant, and not by *selecting a square array* from a given determinant.

It has before been shown that we may pass from theorems concerning gnomonics to those concerning minors, and we have therefore a principle of duality in determinants which may be illustrated by the following dualistic theorems placed side by side.

A determinant P whose elements are p^{th} gnomonics of a determinant A_h of the order h wherein the various gnomons are formed from the rows and columns intersecting in a given determinant A_s of the order s is equal to the original determinant A_h to the $\{(s-1)_p\}^{\text{th}}$ power into the $\{(s-1)_{p-1}\}^{\text{th}}$ power of the determinant obtained by bordering A_h with all the rows and columns intersecting in A_s.	A determinant P' whose elements are p^{th} minors of a determinant A_h of the order h wherein the various aphaeretons are formed from the rows and columns intersecting in a given determinant A_s' of the order s is equal to the original determinant A_h to the $\{(s-1)_p\}^{\text{th}}$ power into the $\{(s-1)_{p-1}\}^{\text{th}}$ power of the determinant obtained by striking out from A_h all the rows and columns intersecting in A_s'.
A minor of P of the order m is equal to the original determinant A_h to the power $m-(s-1)_{p-1}$ into the $\{m-(s-1)_p\}^{\text{th}}$ power of the determinant obtained by bordering A_h with all the rows and columns intersecting in A_s into the complementary minor of the reciprocal determinant.	A minor of P' of the order m is equal to the original determinant A_h to the power $m-(s-1)_{p-1}$ into the $\{m-(s-1)_p\}^{\text{th}}$ power of the determinant obtained by striking out from A_h all the rows and columns intersecting in A_s' into the complementary minor of the reciprocal determinant.
If a determinant vanish the elements in any row of the determinant of first gnomonics are proportional to the elements in any other row.	If a determinant vanish the elements in any row of the determinant of first minors are proportional to the elements in any other row.

Now consider the determinant ad, Fig. 4, divided by the dotted lines in the figure, so that kl is a square array say of the order $k (< h)$. Form two compound determinants whose elements are minors formed by removing aphaeretons of the orders p and $k-p$ respectively, whose rows and columns intersect in kl.

These two determinants are reciprocal, and we may apply to either one the first two theorems given on the right-hand side of the line dividing dualistic theorems. Denote the elements of the first by $p'_{r,s}$, of the second by $q'_{r,s}$, the first compound determinant by P', and the second by Q'. It is evident that the elements $p'_{r,s}$ reduce to those determinants formed by striking out p of the k right-hand columns of ab and substituting in their stead p of the k right-hand columns of bg, while the elements $q'_{r,s}$ reduce to those determinants formed by striking out $k-p$ of the right-hand columns of ab and substituting in their stead $k-p$ of the k right-hand columns of bg, where in forming any element of P' neither striking out nor substituting affects the same columns as in forming the corresponding element of Q'. By applying the first theorem on the right of the line dividing dualistic theorems it follows that

$$P' = (ad)^{(k-1)_p} . (lm)^{(k-1)_{p-1}}.$$

But $ad = A_h$ and lm reduces to the determinant formed by striking out the k right-hand columns of A_h and substituting the k right-hand columns of bg in their place.

Hence the determinant P' equals the original determinant A_h to the power $(k-1)_p$ into the $\{(k-1)_{p-1}\}^{\text{th}}$ power of the determinant produced by striking out and substituting all at once the rows and columns that were struck out and substituted piecemeal in forming the elements of P'.

Now remembering what the elements of P' are we may produce this determinant in another way, as follows: Take two determinants A_h and B_h, placed side by side, and form a determinant whose elements are those determinants obtained by replacing in every possible way p columns from a fixed selection of k columns of A_h, by p columns taken in every possible way from a fixed selection of k columns of B_h. This determinant, P', then equals the original determinant A_h to the power $(k-1)_p$ into the $\{(k-1)_{p-1}\}^{\text{th}}$ power of the determinant produced by replacing the fixed selection of k columns of A_h by the fixed selection of k columns of B_h.

Similarly the determinant Q' equals the determinant B_h to the power $(k-1)_p$ into the $\{(k-1)_{p-1}\}^{\text{th}}$ power of the determinant obtained by replacing the fixed selection of k columns of B_h by the fixed selection of k columns of A_h.

If $k = h$, *i. e.* if the fixed selection of k columns of A_h and B_h is in each case the *whole determinant*, then the determinants P' and Q' are the determinants described in Scott's Determinants, page 56, Art. 4. Moreover, when $k < h$, these determinants P' and Q' are minors of the determinants given in Scott's treatise, but they are here expressed entirely independent of a complementary minor of another determinant.

Applying the second of the theorems on the right of the line dividing dualistic theorems, it readily follows that a minor of P' of the order m is equal to the determinant A_h to the power $m - (k - 1)_{p-1}$ into the $\{m - (k - 1)_p\}^{\text{th}}$ power of the determinant obtained by replacing the fixed selection of k columns of A_h by the fixed selection of k columns of B_h, into the reciprocal determinant Q'.

P. S.—Since writing the above I have received and read an article on Compound Determinants by Mr. R. F. Scott, published in the Proceedings of the London Mathematical Society, Vol. XIV, page 91.

I was never satisfied with Picquet's proof of the theorem discovered by Professor Sylvester and tried to find some simple and rigorous proof. Failing in this, I wrote out the one given above as one to my own mind preferable to Picquet's, but Mr. Scott's proof, in section 2 of his article, seems to leave nothing to be desired either in simplicity or rigor.

In the determinant which I have called A_n, if $n = 2h$, and if A_n is a determinant formed by putting on A_h a gnomon of the order h, wherein the h^2 elements at the intersection of the rows and columns forming the gnomon are all zeros, then the theorems given by Mr. Scott, in his article from section 5 on, are easily seen to be special cases of some of the theorems given in this paper.